MATH 60604A
Statistical modelling
§ 6d - Random intercept model

HEC Montréal
Department of Decision Sciences

Random intercept linear model

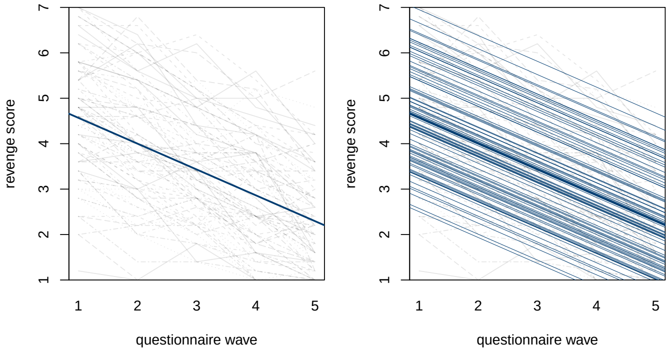
- The simplest random effects model is one with only a group-specific random intercept.
- The equation of the linear mixed model is

$$Y_{ij} = \beta_0 + b_i + \beta_1 \mathrm{X}_{ij1} + \cdots + \beta_p \mathrm{X}_{ijp} + \varepsilon_{ij}, \qquad \boldsymbol{\varepsilon}_i \sim \mathrm{No}(\mathbf{0}, \mathbf{R}_i)$$

  for $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$ and where Y_{ij} is observation j from group i.
- The intercept specific to group i is $\beta_0 + b_i$. It consists of
  - A **common effect over all groups**, β_0;
  - A **group-specific effect**, b_i.

Graphical illustration

Consider a random intercept model for the `revenge` data, with AR(1) errors and `t` as fixed effect.



Model without (left) and with random intercept for `id` (right).

Assumptions of the random intercept model

The model equation is

$$Y_{ij} = (\beta_0 + b_i) + \beta_1 X_{ij1} + \beta_2 X_{ij2} + \cdots + \beta_p X_{ijp} + \varepsilon_{ij}$$

- The random effects $b_1, \ldots, b_m$ are assumed to be independent from the ε terms and the explanatory variables).
- We assume for the time being
  - $b_i \overset{\text{iid}}{\sim} \mathsf{No}(0, \sigma_b^2)$ $(i = 1, \ldots, m)$.
  - $\varepsilon_{ij} \overset{\text{iid}}{\sim} \mathsf{No}(0, \sigma^2)$ $(i = 1, \ldots, m; j = 1, \ldots, n_i)$.

Random effects models: covariance

Since it's random, the term b_i introduces a within-group correlation in the model. Because ε_{ij} is independent of b_i for all i, j, the (conditional) variance of an observation is

$$\mathrm{Var}\left(Y_{ij} \mid \mathbf{X}_i\right) = \mathrm{Var}\left(b_i\right) + \mathrm{Var}\left(\varepsilon_{ij}\right) = \sigma_b^2 + \sigma^2$$

The covariance between two individuals in the same group is

$$\mathrm{Cov}\left(Y_{ij}, Y_{ik} \mid \mathbf{X}_i\right) = \sigma_b^2, \qquad j \neq k.$$

Consequently, the correlation between two individuals in the same group is

$$\mathrm{Corr}\left(Y_{ij}, Y_{ik} \mid \mathbf{X}_i\right) = \frac{\sigma_b^2}{\sigma^2 + \sigma_b^2}, \qquad j \neq k.$$

This quantity is often called the intra-class correlation.

Mathematical aside

Both β_j and explanatories are assumed non-random, thus

$$\begin{aligned}
\mathsf{Cov}\left(Y_{ij}, Y_{ik} \mid \mathbf{X}_i\right) &= \mathsf{Co}\left(\beta_0 + b_i + \beta_1 \mathrm{X}_{ij1} + \cdots + \varepsilon_{ij},\right. \\
&\qquad\qquad \left.\beta_0 + b_i + \beta_1 \mathrm{X}_{ik1} + \cdots + \varepsilon_{ik} \mid \mathbf{X}_i\right) \\
&= \mathsf{Cov}\left(b_i + \varepsilon_{ij}, b_i + \varepsilon_{ik}\right) \\
&= \mathsf{Var}\left(b_i\right) + \mathsf{Cov}\left(\varepsilon_{ij}, \varepsilon_{ik}\right) \\
&= \sigma_b^2 + \sigma^2 \mathbf{1}_{j=k}.
\end{aligned}$$

where the last step follows from independence of b_i and ε's and because $\mathsf{Cov}\left(Y_{ij}, Y_{ij}\right) = \mathsf{Var}\left(Y_{ij}\right)$.

Unconditional variance of Y

Alternatively,

$$\begin{aligned}
\mathsf{Var}\left(Y_i \mid \mathbf{X}_i\right) &= \mathsf{Var}\left(b_i \mathbf{1}_{n_i}\right) + \mathsf{Var}\left(\boldsymbol{\varepsilon}_i\right) \\
&= \sigma_b^2 \mathbf{1}_{n_i} \mathbf{1}_{n_i}^\top + \sigma^2 \mathbf{I}_{n_i} \\
&= \begin{pmatrix} \sigma_b^2 & \sigma_b^2 & \cdots & \sigma_b^2 \\ \sigma_b^2 & \sigma_b^2 & \cdots & \sigma_b^2 \\ \vdots & \ddots & \ddots & \vdots \\ \sigma_b^2 & \sigma_b^2 & \cdots & \sigma_b^2 \end{pmatrix} + \begin{pmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{pmatrix}.
\end{aligned}$$

Compound symmetry correlation induced by the random intercept model

- When the error terms ε_i are independent with $\mathrm{Var}\,(\varepsilon_i) = \sigma^2 \mathbf{I}_{n_i}$, introducing a random effect b_0 for the intercept implies that the within-group correlation is the same.
- In that particular case, the conditional covariance matrix of Y_i is thus the same as if we considered a linear regression model with no random effect and a compound symmetry model for $\mathrm{Var}\,(\varepsilon_i)$.
- The difference is that now the correlation must be non-negative, since σ_b^2 is a variance. This limitation is not usually of consequence, because within-group correlations tend to be positive.

Adding a random intercept with the `random` command

- The command `repeated` allows us to specify the covariance structure for the errors in `proc mixed`.
- If we don't use the `repeated` command, the errors are assumed to be independent.

SAS code for a random intercept model with independent errors

```
proc mixed data=statmod.motivation;
class idunit;
model motiv = sex yrserv agemanager nunit / solution;
random intercept / subject=idunit v=1 vcorr=1;
run;
```

Including a random intercept induces a compound symmetry correlation structure. Therefore, we do not need to specify anything for the covariance structure of the errors.

Covariance matrix specified by the random intercept model

Estimated V Matrix for idunit 1

Row	Col1	Col2	Col3	Col4	Col5	Col6	Col7	Col8	Col9
1	1.3709	0.2448	0.2448	0.2448	0.2448	0.2448	0.2448	0.2448	0.2448
2	0.2448	1.3709	0.2448	0.2448	0.2448	0.2448	0.2448	0.2448	0.2448
3	0.2448	0.2448	1.3709	0.2448	0.2448	0.2448	0.2448	0.2448	0.2448
4	0.2448	0.2448	0.2448	1.3709	0.2448	0.2448	0.2448	0.2448	0.2448
5	0.2448	0.2448	0.2448	0.2448	1.3709	0.2448	0.2448	0.2448	0.2448
6	0.2448	0.2448	0.2448	0.2448	0.2448	1.3709	0.2448	0.2448	0.2448
7	0.2448	0.2448	0.2448	0.2448	0.2448	0.2448	1.3709	0.2448	0.2448
8	0.2448	0.2448	0.2448	0.2448	0.2448	0.2448	0.2448	1.3709	0.2448
9	0.2448	0.2448	0.2448	0.2448	0.2448	0.2448	0.2448	0.2448	1.3709

Covariance parameter estimates

Covariance Parameter Estimates		
Cov Parm	**Subject**	**Estimate**
Intercept	idunit	0.2448
Residual		1.1261

- The variance estimate for the random intercept is $\widehat{\sigma}_b^2 = 0.2448$, whereas the estimate of the variance of the error term is $\widehat{\sigma}^2 = 1.1261$.
- Consequently, the estimate of the within-unit correlation is

$$\widehat{\rho} = \frac{\widehat{\sigma}_b^2}{\widehat{\sigma}_b^2 + \sigma^2} = 0.1785.$$

- This is exactly the same correlation for the observation than that obtained from compound symmetry covariance model for the errors (command `repeated`).

Fixed effect estimates

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr > |t|
Intercept	13.7633	0.3955	97	34.80	<.0001
sex	0.5622	0.06835	914	8.23	<.0001
yrserv	-0.4722	0.006015	914	-78.50	<.0001
agemanager	0.01929	0.006801	97	2.84	0.0056
nunit	0.006470	0.02019	97	0.32	0.7493

The effects of the explanatory variables (and their standard errors) are also the same as for the compound symmetry structure — both models are equivalent for the response assuming the within-unit correlation is positive.