Complete factorial designs

Session 5

MATH 80667A: Experimental Design and Statistical Methods
for Quantitative Research in Management
HEC Montréal

Outline

Factorial designs and interactions

Model formulation

Factorial designs and interactions

Complete factorial designs?

Factorial design

study with multiple factors (subgroups)

Complete

Gather observations for every subgroup

Motivating example

Response:
retention of information
two hours after reading a story

Population:
children aged four

experimental factor 1:
ending (happy or sad)

experimental factor 2:
complexity (easy, average or hard).

Setup of design

complexity	happy	sad
complicated		
average		
easy		

Factors are crossed

Efficiency of factorial design

Cast problem
as a series of one-way ANOVA
vs simultaneous estimation

Factorial designs requires
fewer overall observations

Can study interactions

Interaction

Definition: when the effect of one factor depends on the levels of another factor.

Effect together

$\neq$

sum of individual effects

Interaction or profile plot

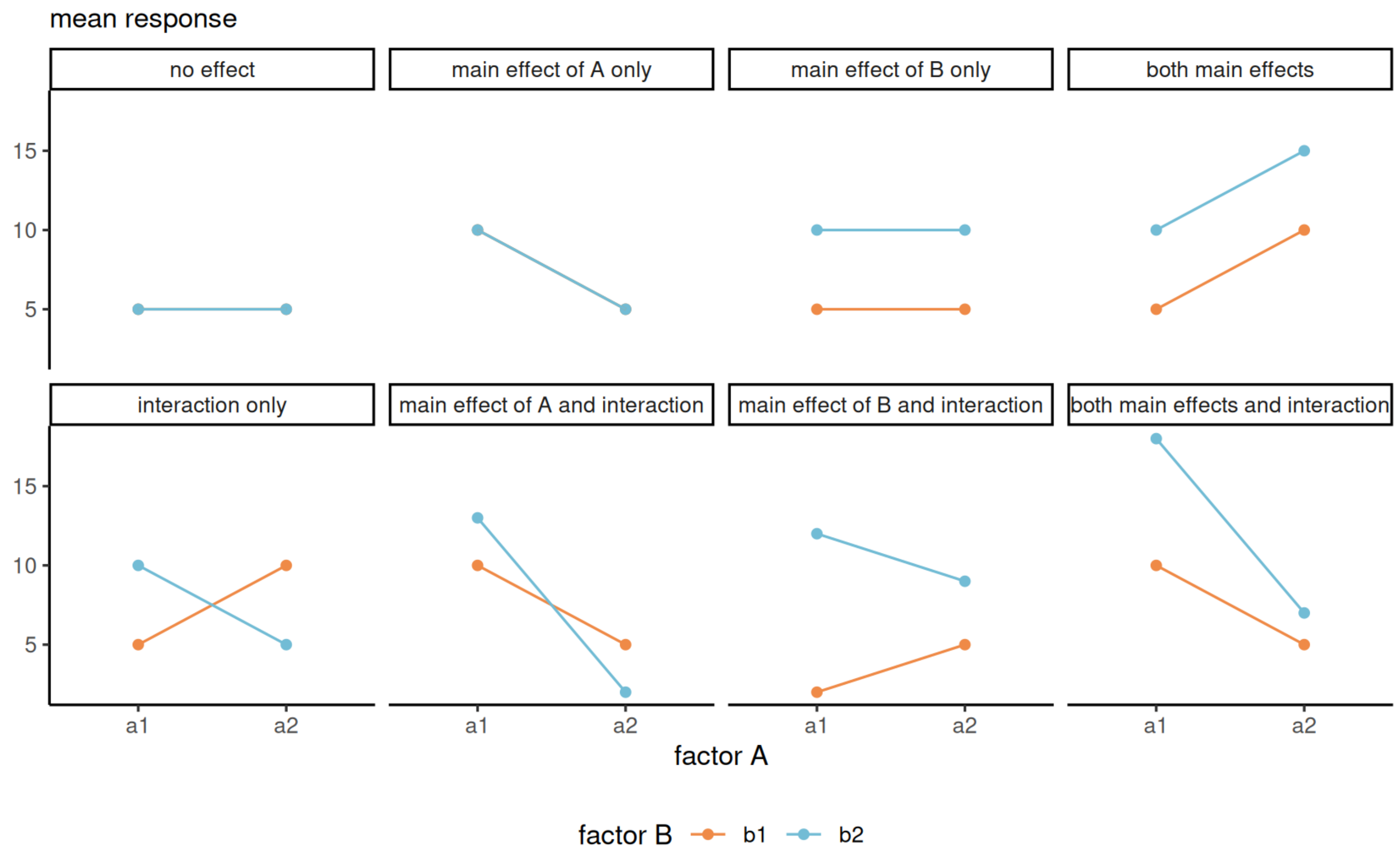
**Graphical display:
plot sample mean per category**

**with uncertainty measure
(1 std. error for mean
confidence interval, etc.)**

Interaction: lines are not parallel

No interaction: parallel lines

Interaction plot for 2 by 2 design



Model formulation

Formulation of the two-way ANOVA

Two factors: A (complexity) and B (ending) with n_a and n_b levels.

Write the average response Y_{ijr} of the rth measurement in group (a_i, b_j) as

$$\underbrace{Y_{ijr}}_{\text{response}} = \underbrace{\mu_{ij}}_{\text{subgroup mean}} + \underbrace{\varepsilon_{ijr}}_{\text{error term}}$$

where

- Y_{ijr} is the rth replicate for ith level of factor A and jth level of factor B
- ε_{ijr} are independent error terms with mean zero and variance σ^2.

One average for each subgroup

B **ending** A **complexity**	b_1 **(happy)**	b_2 **(sad)**	***row mean***
a_1 (complicated)	μ_{11}	μ_{12}	$\mu_{1.}$
a_2 (average)	μ_{21}	μ_{22}	$\mu_{2.}$
a_3 (easy)	μ_{31}	μ_{32}	$\mu_{3.}$
column mean	$\mu_{.1}$	$\mu_{.2}$	μ

Row, column and overall average

- Mean of A_i (average of row i):

$$\mu_{i.} = \frac{\mu_{i1} + \cdots + \mu_{in_b}}{n_b}$$

- Mean of B_j (average of column j):

$$\mu_{.j} = \frac{\mu_{1j} + \cdots + \mu_{n_a j}}{n_a}$$

- Overall average (overall all rows and columns):

$$\mu = \frac{\sum_{i=1}^{n_a} \sum_{j=1}^{n_b} \mu_{ij}}{n_a n_b}$$

Vocabulary of effects

Definitions

- **simple effects**: difference between levels of one in a fixed combination of others (change in difficulty for happy ending)
- **main effects**: differences relative to average for each condition of a factor (happy vs sad ending)
- **interaction effects**: when simple effects differ depending on levels of another factor

What it means relative to the table

- **simple effects** are comparisons between cell averages within a given row or column
- **main effects** are comparisons between row or column averages
- **interaction effects** are difference relative to the row or column average

Marginal effects

	happy	sad
column means	$\mu_{.1}$	$\mu_{.2}$

complexity	row means
complicated	$\mu_{1.}$
average	$\mu_{2.}$
easy	$\mu_{3.}$

Simple effects

	happy	sad
means (easy)	$\mu_{1.}$	$\mu_{2.}$

complexity	mean (happy)
complicated	μ_{11}
average	μ_{21}
easy	μ_{31}

Contrasts

Suppose the order of the coefficients is factor A (complexity, 3 levels, complicated/average/easy) and factor B (ending, 2 levels, happy/sad).

test	μ_{11}	μ_{12}	μ_{21}	μ_{22}	μ_{31}	μ_{32}
main effect A (complicated vs average)	1	1	-1	-1	0	0
main effect A (complicated vs easy)	1	1	0	0	-1	-1
main effect B (happy vs sad)	1	-1	1	-1	1	-1
interaction AB (comp. vs av, happy vs sad)	1	-1	-1	1	0	0
interaction AB (comp. vs easy, happy vs sad)	1	-1	0	0	-1	1

Hypothesis tests for main effects

Generally, need to compare multiple effects at once

Main effect of factor A

$\mathscr{H}_0$: $\mu_{1.} = \cdots = \mu_{n_a.}$ vs $\mathscr{H}_a$: at least two marginal means of A are different

Main effect of factor B

$\mathscr{H}_0$: $\mu_{.1} = \cdots = \mu_{.n_b}$ vs $\mathscr{H}_a$: at least two marginal means of B are different.

Equivalent formulation of the two-way ANOVA

Write the model for a response variable Y in terms of two factors a_i, b_j.

$$Y_{ijr} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijr}$$

where

- $\alpha_i = \mu_{i.} - \mu$
  - mean of level a_i minus overall mean.
- $\beta_j = \mu_{.j} - \mu$
  - mean of level b_j minus overall mean.
- $(\alpha\beta)_{ij} = \mu_{ij} - \mu_{i.} - \mu_{.j} + \mu$
  - the interaction term for a_i and b_j.

One average for each subgroup

B **ending** / A **complexity**	b_1 **(happy)**	b_2 **(sad)**	***row mean***
a_1 (complicated)	$\mu + \alpha_1 + \beta_1 + (\alpha\beta)_{11}$	$\mu + \alpha_1 + \beta_2 + (\alpha\beta)_{12}$	$\mu + \alpha_1$
a_2 (average)	$\mu + \alpha_2 + \beta_1 + (\alpha\beta)_{21}$	$\mu + \alpha_2 + \beta_2 + (\alpha\beta)_{22}$	$\mu + \alpha_2$
a_3 (easy)	$\mu + \alpha_3 + \beta_1 + (\alpha\beta)_{31}$	$\mu + \alpha_3 + \beta_2 + (\alpha\beta)_{32}$	$\mu + \alpha_3$
column mean	$\mu + \beta_1$	$\mu + \beta_2$	μ

More parameters than data cells!

The model in terms of α, β and $(\alpha\beta)$ is overparametrized.

Sum-to-zero parametrization

Too many parameters!

Impose sum to zero constraints

$$\sum_{i=1}^{n_a} \alpha_i = 0, \quad \sum_{j=1}^{n_b} \beta_j = 0, \quad \sum_{j=1}^{n_b} (\alpha\beta)_{ij} = 0, \quad \sum_{i=1}^{n_a} (\alpha\beta)_{ij} = 0.$$

which imposes $1 + n_a + n_b$ constraints.

Why use the sum to zero parametrization?

- Testing for main effect of A:

$$\mathscr{H}_0 : \alpha_1 = \cdots = \alpha_{n_a} = 0$$

- Testing for main effect of B:

$$\mathscr{H}_0 : \beta_1 = \cdots = \beta_{n_b} = 0$$

- Testing for interaction between A and B:

$$\mathscr{H}_0 : (\alpha\beta)_{11} = \cdots = (\alpha\beta)_{n_a n_b} = 0$$

In all cases, alternative is that at least two coefficients are different.

Seeking balance

Balanced sample
(equal nb of obs per group)

With n_r replications per subgroup, total sample size is $n = n_a n_b n_r$.

Why balanced design?

With equal variance, this is the optimal allocation of treatment unit.

maximize power

Estimated means for main and total effects correspond to marginal averages.

equiweighting

Unambiguous decomposition of effects of A, B and interaction.

orthogonality

Rewriting observations

$$\begin{aligned}
\underbrace{(y_{ijr} - \widehat{\mu})}_{\text{obs vs grand mean (total)}} &= \underbrace{(\widehat{\mu}_{i.} - \widehat{\mu})}_{\text{row mean vs grand mean}(A)} \\
&+ \underbrace{(\widehat{\mu}_{.j} - \widehat{\mu})}_{\text{col mean vs grand mean}(B)} \\
&+ \underbrace{(\widehat{\mu}_{ij} - \widehat{\mu}_{i.} - \widehat{\mu}_{.j} + \widehat{\mu})}_{\text{cell mean vs additive effect}(AB)} \\
&+ \underbrace{(y_{ijr} - \widehat{\mu}_{ij})}_{\text{obs vs cell mean (resid)}}
\end{aligned}$$

Decomposing variability

Constructing statistics as before by decomposing variability into blocks.

We can square both sides and sum over all observations.

With balanced design, all cross terms cancel, leaving us with the **sum of square** decomposition

$$\mathsf{SS}_{\text{total}} = \mathsf{SS}_A + \mathsf{SS}_B + \mathsf{SS}_{AB} + \mathsf{SS}_{\text{resid}}.$$

Sum of square decomposition

The sum of square decomposition

$$\mathsf{SS}_{\text{total}} = \mathsf{SS}_A + \mathsf{SS}_B + \mathsf{SS}_{AB} + \mathsf{SS}_{\text{resid}}.$$

is an estimator of the population variance decomposition

$$\sigma^2_{\text{total}} = \sigma^2_A + \sigma^2_B + \sigma^2_{AB} + \sigma^2_{\text{resid}}.$$

where $\sigma^2_A = n_a^{-1} \sum_{i=1}^{n_a} \alpha_i^2$, $\sigma^2_{AB} = (n_a n_b)^{-1} \sum_{i=1}^{n_a} \sum_{j=1}^{n_b} (\alpha\beta)^2_{ij}$, etc.

Take ratio of variability (effect relative to residual) and standardize numerator and denominator to build an F statistic.

Analysis of variance table

term	degrees of freedom	mean square	F
A	$n_a - 1$	$\mathsf{MS}_A = \mathsf{SS}_A/(n_a - 1)$	$\mathsf{MS}_A/\mathsf{MS}_{\text{res}}$
B	$n_b - 1$	$\mathsf{MS}_B = \mathsf{SS}_B/(n_b - 1)$	$\mathsf{MS}_B/\mathsf{MS}_{\text{res}}$
AB	$(n_a - 1)(n_b - 1)$	$\mathsf{MS}_{AB} = \mathsf{SS}_{AB}/\{(n_a - 1)(n_b - 1)\}$	$\mathsf{MS}_{AB}/\mathsf{MS}_{\text{res}}$
residuals	$n - n_a n_b$	$\mathsf{MS}_{\text{resid}} = \mathsf{SS}_{\text{res}}/(n - ab)$	
total	$n - 1$		

Read the table backward (starting with the interaction).

- If there is a significant interaction, the main effects are **not** of interest and potentially misleading.

Intuition behind degrees of freedom

B **ending** A **complexity**	b_1 **(happy)**	b_2 **(sad)**	***row mean***
a_1 (complicated)	μ_{11}	x	$\mu_{1.}$
a_2 (average)	μ_{21}	x	$\mu_{2.}$
a_3 (easy)	x	x	x
column mean	$\mu_{.1}$	x	μ

Terms with x are fully determined by row/column/total averages

Multiplicity correction

With equal sample size and equal variance, usual recipes for ANOVA hold.

Correction depends on the effect: e.g., for factor A, the critical values are

- Bonferroni: $1-\alpha/(2m)$ quantile of $\mathsf{St}(n-n_an_b)$
- Tukey: Studentized range (`qtukey`)
  - level $1-\alpha/2$, n_a groups, $n-n_an_b$ degrees of freedom.
- Scheffé: critical value is $\{(n_a-1)\mathfrak{f}_{1-\alpha}\}^{1/2}$
  - $\mathfrak{f}_{1-\alpha}$ is $1-\alpha$ quantile of $\mathsf{F}(\nu_1=n_a-1,\nu_2=n-n_an_b)$.

Software implementations available in `emmeans` in **R**.

Numerical example